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Normalization of off-shell boundary state, g-function and zeta function regularization

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Abstract
We consider the model in two dimensions with boundary quadratic deformation (BQD), which has been discussed in tachyon condensation. The partition function of this model (BQD) on a cylinder is determined using the method of zeta function regularization. We show that, for closed channel partition function, a subtraction procedure must be introduced in order to reproduce the correct results at conformal points. The boundary entropy (g-function) is determined from the partition function and the off-shell boundary state. We propose and consider a supersymmetric generalization of the BQD model, which includes a boundary fermion mass term, and check the validity of the subtraction procedure.

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1. Introduction

Quantum field theories on manifolds with boundaries have been studied actively in recent years. They play important roles in various areas of physics. In particular, since the proposal of Sen's conjecture on tachyon condensation [1], the boundary string field theory [2, 3] has received considerable attention [4, 5]. With regard to this last development, the two-dimensional off-critical model with boundary quadratic deformation (BQD) has been considered. It describes an off-shell renormalization group flow from the Neumann boundary condition to the Dirichlet boundary condition. The closed channel partition function on a cylinder has been considered by using the method of thermodynamic Bethe ansatz in the long cylinder limit [6]; see, for instance, [7–9] for related calculation.

In this paper, taking the BQD model with more general boundary conditions, we re-examine the partition function by another method without taking the long cylinder limit.

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In order to get a finite expression for the closed channel partition function, we propose a subtraction procedure which follows the zeta function regularization. From the expression of the partition function, the g-function is determined with the help of an off-shell boundary state [6, 10].

To check the validity of our subtraction procedure, we also consider a supersymmetric generalization of the BQD model. The BQD model can be considered as a weak interaction limit of the boundary sine-Gordon model [11]. We propose a supersymmetric generalization of the BQD model (SBQD model) of tachyon condensation [4, 5] as the weak interaction limit of a supersymmetric boundary sine-Gordon model. (For the bosonic counterpart of this statement, see [12, 13].)

Let us briefly recall the supersymmetric sine-Gordon model on a half-line [14, 15]. The following action is conjectured to be integrable [15] (see, in particular, the note added),

$$S = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\sigma^2 \int_{-\infty}^{0} \mathrm{d}\sigma^1 \mathcal{L}_0 + \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\sigma^2 \mathcal{L}_b \tag{1.1}$$

where

$$\begin{aligned}
\mathcal{L}_0 &= \frac{2}{\alpha'}\bar\partial X \partial X + \psi\bar\partial\psi + \tilde\psi\partial\tilde\psi - \frac{m^2}{\lambda^2\alpha'}(\cos\lambda X - 1) + \mathrm{i}m\tilde\psi\psi\cos\frac{\lambda}{2}X \\
\mathcal{L}_b &= a\partial_2 a + \omega_0^{-1}\lambda\rho^{1/2}a(\psi - \mathrm{i}\tilde\psi)\cos\frac{\lambda}{4}(X-\chi_0) \\
&\qquad - \frac{4}{\alpha'}\left(\Lambda\cos\frac{\lambda}{2}(X - X_0) - \rho\right) + \mathrm{i}\tilde\psi\psi|_{\sigma^1=0}.
\end{aligned} \tag{1.2}$$

Here $\omega_0 = \mathrm{e}^{\mathrm{i}\pi/4}$, $z = \sigma^1 + \mathrm{i}\sigma^2$ and $a = a(\sigma_2)$ is an auxiliary fermionic field. The parameters $\rho\ (>0)$ and χ_0 are related to $\Lambda\ (>0)$ and X_0 by the conditions

$$\rho\sin\frac{\lambda}{2}\chi_0 = \Lambda\sin\frac{\lambda}{2}X_0 \qquad \rho\cos\frac{\lambda}{2}\chi_0 = \Lambda\cos\frac{\lambda}{2}X_0 - \frac{2m}{\lambda^2}. \tag{1.3}$$

Let us consider the massless case $m = 0$. Then $\rho = \Lambda$ and $\chi_0 = X_0$, and we find

$$\begin{aligned}
\mathcal{L}_0 &= \frac{2}{\alpha'}\bar\partial X\partial X + \psi\bar\partial\psi + \tilde\psi\partial\tilde\psi \\
\mathcal{L}_b &= a\partial_2 a + \omega_0^{-1}\lambda\Lambda^{1/2}a(\psi - \mathrm{i}\tilde\psi) - \frac{4\Lambda}{\alpha'}\left(\cos\frac{\lambda}{2}(X - X_0) - 1\right) + \mathrm{i}\tilde\psi\psi|_{\sigma^1=0}.
\end{aligned} \tag{1.4}$$

If we take the $\lambda = 0$ limit with $h = \lambda\Lambda^{1/2}$ fixed, then the boundary Lagrangian becomes

$$\mathcal{L}_b = a\partial_2 a + \omega_0^{-1}ha(\psi - \mathrm{i}\tilde\psi) + \frac{h^2}{2\alpha'}(X - X_0)^2 + \mathrm{i}\tilde\psi\psi|_{\sigma^1=0}. \tag{1.5}$$

The bosonic sector is the BQD model and the fermionic sector is the Ising model with the boundary magnetic field [11, 16, 17]. This action differs from that in [5, 12] by the presence of the boundary fermion mass term. Inspired by this result, we propose that the action of SBQD model on a cylinder is given by equation (3.1). As for the Ising model on a cylinder with boundary magnetic fields, the g-functions are calculated by other methods in the long cylinder length limit [16, 17]. By using our subtraction procedure, we determine the g-functions. The results agree with those of [16, 17]. Also, some of the results reproduce those of [8, 9].

This paper is organized as follows. The BQD model on the cylinder is considered in section 2. The SBQD model is examined in section 3. Section 4 is devoted to discussion.

2. BQD model on a cylinder

In two-dimensional Euclidean space, the action of the boundary quadratic deformation (BQD) model on a cylinder is given by

$$S[X] = \int_0^{2\pi r} \mathrm{d}\sigma^2 L \tag{2.1}$$

where

$$L = \frac{1}{4\pi\alpha'}\left[\int_0^l \mathrm{d}\sigma^1 (\partial_a X)^2 + v(X - X_0)^2\big|_{\sigma^1=0} + v'(X - X_0')^2\big|_{\sigma^1=l}\right]. \tag{2.2}$$

We consider the cylinder of the length l and circumference $2\pi r$.

The boundary conditions at $\sigma^1 = 0$ and $\sigma^1 = l$ are of mixed type:

$$\partial_1 X - v(X - X_0)|_{\sigma^1=0} = 0 \qquad \partial_1 X + v'(X - X_0')|_{\sigma^1=l} = 0. \tag{2.3}$$

We denote the mixed-type boundary condition by $B = B(u)$ where $u = rv$. Also, we denote the Dirichlet-type boundary condition by $D = B(\infty)$ and the Neumann-type boundary condition by $N = B(0)$.

We can expand

$$X(\sigma^1, \sigma^2) = \hat{X}_0(\sigma^1) + \tilde{X}(\sigma^1, \sigma^2) \tag{2.4}$$

where the zero eigenvalue function is given by

$$\hat{X}_0(\sigma^1) = \frac{1}{w + w' + ww'}[ww'(X_0' - X_0)(\sigma^1/l) + (w + ww')X_0 + w'X_0']. \tag{2.5}$$

Here $w = vl$ and $w' = v'l$. By substituting the expansion into the Lagrangian, we have

$$L = \frac{1}{4\pi\alpha'}\int_0^l \mathrm{d}\sigma^1\left[(\partial_2\tilde{X})^2 + \tilde{X}\left(-\partial_1^2\right)\tilde{X}\right] + \frac{1}{4\pi\alpha' l}\frac{ww'}{(w + w' + ww')}(X_0 - X_0')^2. \tag{2.6}$$

The oscillating modes are expressed as

$$\tilde{X}(\sigma^1, \sigma^2) = \sum_{j=1}^{\infty} X_j(\sigma^2) f_j(\sigma^1) \tag{2.7}$$

where

$$\begin{aligned} f_j(\sigma^1) &= (-1)^j\sqrt{\frac{\pi\alpha'}{l\rho(\nu_j)}}\left[\frac{(\pi\nu_j - \mathrm{i}w)}{\sqrt{(\pi\nu_j)^2 + w^2}}\,\mathrm{e}^{\mathrm{i}\pi\nu_j\sigma^1/l} + \frac{(\pi\nu_j + \mathrm{i}w)}{\sqrt{(\pi\nu_j)^2 + w^2}}\,\mathrm{e}^{-\mathrm{i}\pi\nu_j\sigma^1/l}\right] \\ &= -\sqrt{\frac{\pi\alpha'}{l\rho(\nu_j)}}\left[\frac{(\pi\nu_j + \mathrm{i}w')}{\sqrt{(\pi\nu_j)^2 + (w')^2}}\,\mathrm{e}^{\mathrm{i}\pi\nu_j(\sigma^1 - l)/l} + \frac{(\pi\nu_j - \mathrm{i}w')}{\sqrt{(\pi\nu_j)^2 + (w')^2}}\,\mathrm{e}^{-\mathrm{i}\pi\nu_j(\sigma^1 - l)/l}\right]. \end{aligned} \tag{2.8}$$

Here the density function is given by

$$\rho(k) = 1 + \frac{w}{\pi^2k^2 + w^2} + \frac{w'}{\pi^2k^2 + (w')^2} \tag{2.9}$$

and the constants ν_j are the positive solutions of the following relation [9]:

$$\mathrm{e}^{2\pi\mathrm{i}\nu_j}\frac{(\pi\nu_j - \mathrm{i}w)}{(\pi\nu_j + \mathrm{i}w)}\frac{(\pi\nu_j - \mathrm{i}w')}{(\pi\nu_j + \mathrm{i}w')} = 1. \tag{2.10}$$

Here the normalization is chosen such that

$$\frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma^1 f_j(\sigma^1) f_k(\sigma^1) = \delta_{jk} \tag{2.11}$$

and we have arranged the solutions in increasing order:

$$0 \leqslant \nu_1 < \nu_2 < \cdots < \nu_j < \nu_{j+1} < \cdots. \tag{2.12}$$

Then

$$L = \sum_{j=1}^{\infty} \frac{1}{2}\left[(\partial_2 X_j)^2 + \left(\frac{\pi\nu_j}{l}\right)^2 X_j^2\right] + \frac{1}{4\pi\alpha' l}\frac{ww'}{(w + w' + ww')}(X_0 - X_0')^2. \tag{2.13}$$

2.1. Some properties of $\{\nu_j\}$

For later convenience, let us rewrite equation (2.10) in the following form,

$$F_{BB'}^{(-)}(\nu_j) = 0 \tag{2.14}$$

where

$$\begin{aligned} F_{BB'}^{(-)}(k) &= \frac{1}{2}\left[1 - \mathrm{e}^{2\pi \mathrm{i}k}\left(\frac{\pi k - \mathrm{i}w}{\pi k + \mathrm{i}w}\right)\left(\frac{\pi k - \mathrm{i}w'}{\pi k + \mathrm{i}w'}\right)\right] \\ &= \frac{\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\pi k}{(\pi k + \mathrm{i}w)(\pi k + \mathrm{i}w')}\left[(w + w')\cos(\pi k) + (ww' - \pi^2 k^2)\frac{\sin(\pi k)}{\pi k}\right]. \end{aligned} \tag{2.15}$$

Here $B = B(u)$ and $B' = B(u')$, $u' = rv'$.

Let us examine the following entire function:

$$F_-(k) := \left[(w + w')\cos(\pi k) + (ww' - \pi^2 k^2)\frac{\sin(\pi k)}{\pi k}\right]. \tag{2.16}$$

It is an even function and has zeros at $k = \pm\nu_j$, $F_-(0) = (w + w' + ww')$. The key observation is that it can be written as an infinite product:

$$F_-(k) = (w + w' + ww')\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{\nu_j^2}\right]. \tag{2.17}$$

Therefore,

$$\begin{aligned} F_{BB'}^{(-)}(k) &= \frac{\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\pi k}{(\pi k + \mathrm{i}w)(\pi k + \mathrm{i}w')}\left[(w + w')\cos(\pi k) - \{(\pi k)^2 - ww'\}\frac{\sin(\pi k)}{\pi k}\right] \\ &= \frac{\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\pi k(w + w' + ww')}{(\pi k + \mathrm{i}w)(\pi k + \mathrm{i}w')}\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{\nu_j^2}\right]. \end{aligned} \tag{2.18}$$

At conformal points, ν_j can be written explicitly.

(i) For *DD* boundary condition ($w = \infty$ and $w' = \infty$),

$$F_{DD}^{(-)}(k) = \frac{1}{2}(1 - \mathrm{e}^{2\pi k}) = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\sin(\pi k) = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\pi k\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{j^2}\right] \tag{2.19}$$

$$\nu_j = j \quad j = 1, 2, \ldots. \tag{2.20}$$

(ii) For *DN* ($w = \infty$ and $w' = 0$) or *ND* ($w = 0$ and $w' = \infty$),

$$F_{DN}^{(-)}(k) = F_{ND}^{(-)}(k) = \frac{1}{2}(1 + \mathrm{e}^{2\pi k}) = \mathrm{e}^{\pi \mathrm{i}k}\cos(\pi k) = \mathrm{e}^{\mathrm{i}\pi k}\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{(j - 1/2)^2}\right] \tag{2.21}$$

$$\nu_j = j - \tfrac{1}{2} \quad j = 1, 2, \ldots. \tag{2.22}$$

(iii) For NN ($w = 0$ and $w' = 0$),

$$F_{NN}^{(-)}(k) = \frac{1}{2}(1 - \mathrm{e}^{2\pi \mathrm{i}k}) = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\sin(\pi k) = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\pi k \prod_{j=2}^{\infty}\left[1 - \frac{k^2}{(j-1)^2}\right] \tag{2.23}$$

$$\nu_j = j - 1 \quad j = 1, 2, \ldots. \tag{2.24}$$

Other limits are

$$F_{BD}^{(-)}(k) = \frac{1}{2}\left[1 + \mathrm{e}^{2\pi \mathrm{i}k}\left(\frac{\pi k - \mathrm{i}w}{\pi k + \mathrm{i}w}\right)\right] = \frac{\mathrm{e}^{\mathrm{i}\pi k}\pi k(1+w)}{(\pi k + \mathrm{i}w)}\left[\cos(\pi k) + w\frac{\sin(\pi k)}{\pi k}\right]$$
$$= \frac{\mathrm{e}^{\mathrm{i}\pi k}\pi k(1+w)}{(\pi k + \mathrm{i}w)}\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{\nu_j^2}\right]. \tag{2.25}$$

$$F_{BN}^{(-)}(k) = \frac{1}{2}\left[1 - \mathrm{e}^{2\pi \mathrm{i}k}\left(\frac{\pi k - \mathrm{i}w}{\pi k + \mathrm{i}w}\right)\right] = \frac{\mathrm{e}^{\mathrm{i}\pi k}\,\mathrm{i}w}{(\pi k + \mathrm{i}w)}\left[\cos(\pi k) - \frac{\pi k}{w}\sin(\pi k)\right]$$
$$= \frac{\mathrm{e}^{\mathrm{i}\pi k}\,\mathrm{i}w}{(\pi k + \mathrm{i}w)}\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{\nu_j^2}\right]. \tag{2.26}$$

Comparing expressions (2.18) and (2.23), cancellation of two poles and two zeros should occur for $w \to 0$ and $w' \to 0$ limit:

$$\lim_{w,w'\to 0}\frac{(w + w' + ww')}{(\pi k + \mathrm{i}w)(\pi k + \mathrm{i}w')}\left[1 - \frac{k^2}{\nu_1^2}\right] = -1. \tag{2.27}$$

Thus for small w and w', the smallest solution ν_1 approaches zero as

$$\pi\nu_1 \sim \sqrt{w + w' + ww'} \sim \sqrt{w + w'}. \tag{2.28}$$

2.2. The path integral approach

Let us evaluate the partition function

$$Z_{BB'} = \int [\mathrm{d}X]\,\mathrm{e}^{(-S[X])}. \tag{2.29}$$

We can quantize the Lagrangian (2.13) by the standard method of functional integrations (see e.g. appendix A of [18]). We adopt a different approach which will be convenient for our purposes. In this approach, we impose the periodic boundary condition in the σ^2-direction and we further expand the oscillating modes as

$$X_j(\sigma^2) = \sum_{m\in\mathbb{Z}} \mathcal{A}_{m,j} c_m(\sigma^2) \tag{2.30}$$

where $c_0(\sigma^2) = (2\pi r)^{-1/2}$, and $c_m(\sigma^2) = (\pi r)^{-1/2}\sin(m\sigma^2/r)$, $c_{-m}(\sigma^2) = (\pi r)^{-1/2}\cos(m\sigma^2/r)$ for $m > 0$.

Note that

$$\int_0^{2\pi r} \mathrm{d}\sigma^2 c_m(\sigma^2)c_n(\sigma^2) = \delta_{mn} \qquad m, n \in \mathbb{Z}. \tag{2.31}$$

The action S becomes

$$S = S_{\mathrm{cl}} + \sum_{m\in\mathbb{Z}}\sum_{j=1}^{\infty}\frac{1}{2}\left[\left(\frac{m}{r}\right)^2 + \left(\frac{\pi\nu_j}{l}\right)^2\right](\mathcal{A}_{m,j})^2 \tag{2.32}$$

where

$$S_{\rm cl} = \frac{\tau_2}{2\pi\alpha'}\frac{ww'}{w+w'+ww'}(X_0 - X_0')^2 \qquad \tau_2 = \pi r/l. \tag{2.33}$$

Then the partition function is given by

$$Z_{BB'}(X_0 - X_0') = {\rm e}^{-S_{\rm cl}} Z_{BB'}(0) \tag{2.34}$$

where

$$Z_{BB'}(0) = {\rm Det}^{-1/2}\left(-\partial_1^2 - \partial_2^2\right) = \left(\prod_{m\in\mathbb{Z}}\prod_{j=1}^{\infty}\left[(m/r)^2 + (\pi\nu_j/l)^2\right]\right)^{-1/2}. \tag{2.35}$$

Formally, this divergent quantity can be written as

$$Z_{BB'}(0) = \prod_{j=1}^{\infty}\left(\frac{l}{\pi\nu_j}\right)\cdot\prod_{m=1}^{\infty}\prod_{j=1}^{\infty}\left[(m/r)^2 + (\pi\nu_j/l)^2\right]^{-1}. \tag{2.36}$$

We apply the zeta function regularization naively for each infinite product. We regularize the partition function in two steps. Applying the zeta function regularization in different order, we will obtain two equivalent expressions for the partition function.

First, we start from the m-product. We regularize

$$\prod_{m=1}^{\infty}\left[(m/r)^2 + (\pi\nu_j/l)^2\right] = \prod_{m=1}^{\infty}\left(\frac{m}{r}\right)^2\cdot\prod_{m=1}^{\infty}\left[1 + \left(\frac{\tau_2\nu_j}{m}\right)^2\right] \tag{2.37}$$

into

$$\left(\frac{l}{\pi\nu_j}\right) q^{-\nu_j/2}(1 - q^{\nu_j}) \tag{2.38}$$

where $q = {\rm e}^{-2\pi\tau_2}$ and $\tau_2 = \pi r/l$. Then we get the still divergent expression of the partition function:

$$Z_{BB'}(0) = \prod_{j=1}^{\infty} q^{\nu_j/2}(1 - q^{\nu_j})^{-1}. \tag{2.39}$$

The divergence comes from the infinite Casimir energy.

We regularize the Casimir energy and obtain the first regularized expression,

$$Z_{BB'}^{(1)}(0) = q^{(1/2)c_{BB'}^{(-)}}\prod_{j=1}^{\infty}(1 - q^{\nu_j})^{-1} \tag{2.40}$$

where

$$c_{BB'}^{(-)} = \int_0^{\infty}{\rm d}t\, 2t\,\rho({\rm i}t)\left[1 - {\rm e}^{2\pi t}\left(\frac{\pi t + w}{\pi t - w}\right)\left(\frac{\pi t + w'}{\pi t - w'}\right)\right]^{-1}. \tag{2.41}$$

See appendix for details.

Secondly, we regularize the partition function from the j-product. Note that

$$\prod_{j=1}^{\infty}\left[(m/r)^2 + (\pi\nu_j/l)^2\right] = \prod_{j=1}^{\infty}\left(\frac{\pi\nu_j}{l}\right)^2\cdot\prod_{j=1}^{\infty}\left[1 + \frac{(\tilde{\tau}_2 m)^2}{\nu_j^2}\right] \tag{2.42}$$

where $\tilde{\tau}_2 = l/\pi r = 1/\tau_2$. Substituting $k = {\rm i}\tilde{\tau}_2 m$ into (2.18), we have an identity

$$\prod_{j=1}^{\infty}\left[1 + \frac{(\tilde{\tau}_2 m)^2}{\nu_j^2}\right] = \frac{(m+u)(m+u')}{2m(u+u'+\pi\tilde{\tau}_2 uu')}\tilde{q}^{-(1/2)m}\left[1 - \left(\frac{m-u}{m+u}\right)\left(\frac{m-u'}{m+u'}\right)\tilde{q}^m\right] \tag{2.43}$$

where $u = rv$, $u' = rv'$ and $\tilde{q} = \mathrm{e}^{-2\pi\tilde{\tau}_2}$. We need to regularize the following infinite product:

$$G(u,u') := \prod_{m=1}^{\infty} \frac{(m+u)(m+u')}{2m(u+u'+\pi\tilde{\tau}_2 uu')} = \frac{\mathrm{e}^{-\gamma(u+u')}}{\Gamma(u+1)\Gamma(u'+1)} \prod_{m=1}^{\infty} \left\{ \frac{m\,\mathrm{e}^{(u+u')/m}}{2(u+u'+\pi\tilde{\tau}_2 uu')} \right\}. \tag{2.44}$$

Here γ is Euler's constant.

We regularize the divergent sum as

$$\sum_{m=1}^{\infty} \frac{u}{m} \to \sum_{m=1}^{\infty} \left(\frac{u}{m}\right)^s = u^s \zeta(s) \qquad \mathrm{Re}\, s > 1. \tag{2.45}$$

As s approaches 1, the regularized quantity behaves as

$$\begin{aligned} u^s\zeta(s) &= \frac{u}{s-1} + \gamma u + u\log u + \mathcal{O}(s-1) \\ &= \frac{su}{s-1} + \gamma u + u\log u - u + \mathcal{O}(s-1). \end{aligned} \tag{2.46}$$

Our subtraction procedure is to replace the divergent sum $\sum(u/m)$ by

$$\lim_{s\to 1}\left(u^s\zeta(s) - \frac{su}{s-1}\right) = \gamma u + u\log u - u. \tag{2.47}$$

Then the resulting expression is given by

$$G(u,u') = \frac{\sqrt{4\pi}(u+u'+\pi\tilde{\tau}_2 uu')^{1/2}}{\Gamma(u+1)\Gamma(u'+1)} \left(\frac{u}{e}\right)^u \left(\frac{u'}{e}\right)^{u'}. \tag{2.48}$$

The subtraction part is chosen such that the expression has a consistent zeta function regularization relation for large u, u'. For example, $G(\infty,\infty) = \prod_{m=1}^{\infty}(2\pi\tilde{\tau}_2 m)^{-1} = (\tilde{\tau}_2)^{1/2}$.

Then, finally we have another expression,

$$\begin{aligned} Z^{(2)}_{BB'}(0) &= \frac{1}{\sqrt{4\pi}} \frac{\Gamma(1+u)\Gamma(1+u')}{(u+u'+\pi\tilde{\tau}_2 uu')^{1/2}} \left(\frac{e}{u}\right)^u \left(\frac{e}{u'}\right)^{u'} \\ &\quad\times \tilde{q}^{-1/24} \prod_{m=1}^{\infty}\left(1 - \left(\frac{m-u}{m+u}\right)\left(\frac{m-u'}{m+u'}\right)\tilde{q}^m\right)^{-1}. \end{aligned} \tag{2.49}$$

Therefore we conjecture the equality of the two expressions of $Z_{BB'}(0)$, (2.40) and (2.49), $Z^{(1)}_{BB'}(0) = Z^{(2)}_{BB'}(0)$. This is consistent with [8, 9].

We can check our conjecture at conformal points. Indeed,

$$Z^{(1)}_{DD}(0) = q^{-1/24}\prod_{n=1}^{\infty}(1-q^n)^{-1} \qquad Z^{(2)}_{DD}(0) = \frac{1}{\sqrt{\tilde{\tau}_2}}\tilde{q}^{-1/24}\prod_{n=1}^{\infty}(1-\tilde{q}^n)^{-1} \tag{2.50}$$

$$Z^{(1)}_{ND}(0) = q^{1/48}\prod_{n=1}^{\infty}\left(1-q^{(n-1/2)}\right)^{-1} \qquad Z^{(2)}_{ND}(0) = \frac{1}{\sqrt{2}}\tilde{q}^{-1/24}\prod_{n=1}^{\infty}(1+\tilde{q}^n)^{-1}. \tag{2.51}$$

For the NN case ν_1 becomes zero, so we set

$$\hat{Z}^{(i)}_{NN}(0) := \lim_{u,u'\to 0}(1-q^{\nu_1})Z^{(i)}_{BB'}(0) \qquad i = 1, 2. \tag{2.52}$$

Using equation (2.28), we have

$$\hat{Z}^{(1)}_{NN}(0) = q^{-1/24}\prod_{n=1}^{\infty}(1-q^n)^{-1} \qquad \hat{Z}^{(2)}_{NN}(0) = \frac{1}{\sqrt{\tilde{\tau}_2}}\tilde{q}^{-1/24}\prod_{n=1}^{\infty}(1-\tilde{q}^n)^{-1}. \tag{2.53}$$

Thus we see that in the limiting cases we have obtained the correct results: $Z^{(1)}_{DD}(0) = Z^{(2)}_{DD}(0)$, $Z^{(1)}_{ND}(0) = Z^{(2)}_{ND}(0)$, $\hat{Z}^{(1)}_{NN}(0) = \hat{Z}^{(2)}_{NN}(0)$.

2.3. Boundary states and g-function

Let us express $Z_{BB'}$ by using boundary states:

$$(\partial_1 X - v(X - X_0))|_{\sigma^1=0}|B(u); X_0\rangle = 0. \tag{2.54}$$

The mode expansion of X is given by

$$X(\sigma^1, \sigma^2) = x - \frac{\mathrm{i}\alpha'}{r} p\sigma^1 + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n\, \mathrm{e}^{-n(\sigma^1+\mathrm{i}\sigma^2)/r} + \tilde{\alpha}_n\, \mathrm{e}^{-n(\sigma^1-\mathrm{i}\sigma^2)/r}\right) \tag{2.55}$$

and the commutation relations are

$$[x, p] = \mathrm{i} \qquad [\alpha_m, \alpha_n] = m\delta_{m+n,0} \qquad [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n} \qquad [\alpha_m, \tilde{\alpha}_n] = 0. \tag{2.56}$$

Our normalization for the momentum and position eigenstates is

$$\langle k|k'\rangle = \delta(k - k') \qquad \langle X|X'\rangle = \delta(X - X') \qquad \langle X|k\rangle = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{\mathrm{i}kX}. \tag{2.57}$$

The oscillator vacuum $|0\rangle_\alpha$ is normalized as ${}_\alpha\langle 0|0\rangle_\alpha = 1$. The vacuum state is denoted by $|0\rangle = |k = 0\rangle \otimes |0\rangle_\alpha$. The Hamiltonian of the system is

$$H = \frac{1}{r}\left(L_0 + \tilde{L}_0 - \frac{1}{12}\right) \tag{2.58}$$

where

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{n>0} \alpha_{-n}\alpha_n \qquad \tilde{L}_0 = \frac{\alpha'}{4} p^2 + \sum_{n>0} \tilde{\alpha}_{-n}\tilde{\alpha}_n. \tag{2.59}$$

The boundary state is given by

$$|B(u); X_0\rangle = \mathcal{N}(u) \exp\!\left(-\frac{u}{2\alpha'}(x - X_0)^2\right) \exp\!\left(-\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n-u}{n+u}\right) \tilde{\alpha}_{-n}\alpha_{-n}\right) |0\rangle \tag{2.60}$$

where

$$\mathcal{N}(u) = (2\alpha')^{-1/4} \left(\frac{e}{u}\right)^u \Gamma(1 + u). \tag{2.61}$$

The normalization $\mathcal{N}(u)$ is fixed by requiring

$$Z^{(2)}_{BB'}(X_0 - X'_0) = \langle B(u'); X'_0|\, \mathrm{e}^{-lH} |B(u); X_0\rangle. \tag{2.62}$$

The Neumann boundary state is

$$|N\rangle = \lim_{u\to 0} |B(u); X_0\rangle = (2\alpha')^{-1/4} \exp\!\left(-\sum_{n=1}^{\infty} \frac{1}{n}\tilde{\alpha}_{-n}\alpha_{-n}\right) |0\rangle. \tag{2.63}$$

The Dirichlet boundary state is

$$|D; X_0\rangle = \lim_{u\to\infty} |B(u); X_0\rangle = (2\pi^2\alpha')^{1/4} \exp(-\mathrm{i}X_0 p) \exp\!\left(\sum_{n=1}^{\infty} \frac{1}{n}\tilde{\alpha}_{-n}\alpha_{-n}\right) |0\rangle\rangle. \tag{2.64}$$

Here $|0\rangle\rangle = |X = 0\rangle \otimes |0\rangle_\alpha$.

It is easy to evaluate the g-function

$$g(u) := \langle 0|B(u); X_0\rangle = (\alpha'/2)^{1/4} (2\pi u)^{-1/2} \Gamma(u + 1)(e/u)^u. \tag{2.65}$$

In the IR limit, $g_{\mathrm{IR}} = g(\infty) = \langle 0|D; X_0\rangle = (\alpha'/2)^{1/4}$, and in the UV limit, $g_{\mathrm{UV}} = g(0) = \langle 0|N\rangle = (2\alpha')^{-1/4}\, \delta(0) = \infty$. We see that $g_{\mathrm{UV}}/g_{\mathrm{IR}} = (\alpha')^{-1/2}\delta(0) = \infty$. This is consistent with the g-theorem [19].

3. SBQD model on a cylinder

We propose that the action of the supersymmetric BQD model on the cylinder is given by

$$S = \frac{1}{2\pi}\int_0^{2\pi r} d\sigma^2 \int_0^l d\sigma^1 \mathcal{L}_0 + \frac{1}{4\pi}\int_0^{2\pi r} d\sigma^2 \mathcal{L}_b \tag{3.1}$$

where

$$\begin{aligned}\mathcal{L}_0 &= \frac{2}{\alpha'}\partial_{\bar z} X \partial_z X + \psi\partial_{\bar z}\psi + \tilde\psi\partial_z\tilde\psi \\ \mathcal{L}_b &= a_1\partial_2 a_1 + h\omega_0^{-1}a_1(\psi + i\tilde\psi) + \frac{1}{2\alpha'}h^2(X - X_0)^2 + i\tilde\psi\psi|_{\sigma^1=0} \\ &\quad + a_2\partial_2 a_2 + h'\omega_0^{-1}a_2(\psi - i\tilde\psi) + \frac{1}{2\alpha'}(h')^2(X - X_0')^2 + i\tilde\psi\psi|_{\sigma^1=l}.\end{aligned} \tag{3.2}$$

Here $\omega_0 = e^{i\pi/4}$ and $h, h' \in \mathbb{R}$.

The action is a direct sum of a bosonic part and a fermionic part: $S = S[X] + S[\psi, \tilde\psi]$. Thus the partition function of the SBQD model factorizes as $Z = Z^{(X)}_{BB'} Z^{(\psi)}(h, h')$.

From this action, we have the following boundary conditions:

$$\partial_1 X - \tfrac{1}{2}h^2(X - X_0)|_{\sigma^1=0} = 0 \qquad \partial_1 X + \tfrac{1}{2}(h')^2(X - X_0')|_{\sigma^1=l} = 0 \tag{3.3}$$

$$\psi - i\tilde\psi + h\omega_0^{-1}a_1|_{\sigma^1=0} = 0 \qquad \psi + i\tilde\psi + h'\omega_0^{-1}a_2|_{\sigma^1=l} = 0 \tag{3.4}$$

$$2\partial_2 a_1 - h\omega_0^{-1}(\psi + i\tilde\psi)|_{\sigma^1=0} = 0 \qquad 2\partial_2 a_2 + h'\omega_0^{-1}(\psi - i\tilde\psi)|_{\sigma^1=l} = 0. \tag{3.5}$$

After integrating over the auxiliary fermionic fields $a_1(\sigma^2)$ and $a_2(\sigma^2)$, the boundary conditions become

$$\partial_1 X - v(X - X_0)|_{\sigma^1=0} = 0 \qquad \partial_1 X + v'(X - X_0')|_{\sigma^1=l} = 0 \tag{3.6}$$

$$(\partial_2 - iv)\psi|_{\sigma^1=0} = i(\partial_2 + iv)\tilde\psi|_{\sigma^1=0} \qquad (\partial_2 + iv')\psi|_{\sigma^1=l} = -i(\partial_2 - iv')\tilde\psi|_{\sigma^1=l}. \tag{3.7}$$

Here $v = \frac{1}{2}h^2$, $v' = \frac{1}{2}(h')^2$. With this identification of parameters, the bosonic part is the same as that of the previous section.

From now on, we consider the fermionic part only. Let us expand

$$\psi(z) = \sum_l b_l\, e^{ik_l z} \qquad \tilde\psi(\bar z) = \sum_l \tilde b_l\, e^{-ik_l \bar z}. \tag{3.8}$$

Then

$$\tilde b_l = -i\left(\frac{k_l + iv}{k_l - iv}\right) b_l = i\left(\frac{k_l - iv'}{k_l + iv'}\right) e^{2ik_l l} b_l. \tag{3.9}$$

Thus the momentum k_l must satisfy

$$e^{2ik_l l}\left(\frac{k_l - iv}{k_l + iv}\right)\left(\frac{k_l - iv'}{k_l + iv'}\right) + 1 = 0. \tag{3.10}$$

3.1. Spectral determining function

Let $\{\lambda_j\}$ be a set of positive solutions of $F^{(+)}_{BB'}(k) = 0$ where

$$F^{(+)}_{BB'}(k) = \frac{1}{2}\left[1 + e^{2\pi ik}\left(\frac{\pi k - iw}{\pi k + iw}\right)\left(\frac{\pi k - iw'}{\pi k + iw'}\right)\right]. \tag{3.11}$$

This function has two poles at $\pi k = -\mathrm{i}w$ and $\pi k = -\mathrm{i}w'$ and an infinite number of zeros at $k = \pm\lambda_j$. We can set the order of λ_j as

$$0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_j < \lambda_{j+1} < \cdots. \tag{3.12}$$

The function (3.11) can be rewritten as

$$\begin{aligned} F^{(+)}_{BB'}(k) &= \frac{\mathrm{e}^{\pi \mathrm{i}k}}{(\pi k + \mathrm{i}w)(\pi k + \mathrm{i}w')}[(\pi^2 k^2 - ww')\cos(\pi k) + (w + w')\pi k \sin(\pi k)] \\ &= \frac{\mathrm{e}^{\pi \mathrm{i}k}}{(1 - \mathrm{i}(\pi k/w))(1 - \mathrm{i}(\pi k/w'))} \prod_{j=1}^{\infty} \left[1 - \frac{k^2}{\lambda_j^2}\right]. \end{aligned} \tag{3.13}$$

(i) For DD ($w \to \infty$ and $w' \to \infty$),

$$F^{(+)}_{DD}(k) = \frac{1}{2}(1 + \mathrm{e}^{2\pi \mathrm{i}k}) = \mathrm{e}^{\pi \mathrm{i}k}\cos\pi k = \mathrm{e}^{\pi \mathrm{i}k}\prod_{j=1}^{\infty}\left(1 - \frac{k^2}{(j-1/2)^2}\right) \tag{3.14}$$

$$\lambda_j = j - \tfrac{1}{2} \quad j = 1, 2, \ldots. \tag{3.15}$$

(ii) For DN or ND ($w \to \infty$ and $w' \to 0$, or $w \to 0$ and $w' \to \infty$),

$$F^{(+)}_{DN}(k) = F^{(+)}_{ND}(k) = \frac{1}{2}(1 - \mathrm{e}^{2\pi \mathrm{i}k}) = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\sin\pi k = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}k}\pi k\prod_{j=2}^{\infty}\left(1 - \frac{k^2}{(j-1)^2}\right) \tag{3.16}$$

$$\lambda_1 = 0 \qquad \lambda_j = j - 1 \qquad j = 2, 3, \ldots. \tag{3.17}$$

(iii) For NN ($w \to 0$ and $w' \to 0$),

$$F^{(+)}_{NN}(k) = \frac{1}{2}(1 + \mathrm{e}^{2\pi \mathrm{i}k}) = \mathrm{e}^{\pi \mathrm{i}k}\cos\pi k = \mathrm{e}^{\pi \mathrm{i}k}\prod_{j=2}^{\infty}\left(1 - \frac{k^2}{(j-3/2)^2}\right) \tag{3.18}$$

$$\lambda_j = j - \tfrac{3}{2} \quad j = 2, 3, \ldots. \tag{3.19}$$

There is a subtlety in the NN limit. Note that

$$\lim_{w,w' \to 0} \lambda_1 = 0 \tag{3.20}$$

$$0 = \lim_{w,w' \to 0} F^{(+)}_{BB'}(\lambda_1) \neq \lim_{w,w' \to 0} F^{(+)}_{BB'}(0) = F^{(+)}_{NN}(0) = 1. \tag{3.21}$$

Although $F^{(+)}_{NN}(0) \neq 0$, it is convenient to set $\lambda_1 = 0$ at the NN point.

The other limits are

$$\begin{aligned} F^{(+)}_{BD}(k) &= \frac{1}{2}\left[1 - \mathrm{e}^{2\pi \mathrm{i}k}\left(\frac{\pi k - \mathrm{i}w}{\pi k + \mathrm{i}w}\right)\right] = \frac{\mathrm{e}^{\pi \mathrm{i}k}}{(1 - \mathrm{i}(\pi k/w))}\left[\cos(\pi k) - \frac{\pi k}{w}\sin(\pi k)\right] \\ &= \frac{\mathrm{e}^{\pi \mathrm{i}k}}{(1 - \mathrm{i}(\pi k/w))}\prod_{j=1}^{\infty}\left[1 - \frac{k^2}{\lambda_j^2}\right] \end{aligned} \tag{3.22}$$

$$\begin{aligned} F^{(+)}_{BN}(k) &= \frac{1}{2}\left[1 + \mathrm{e}^{2\pi \mathrm{i}k}\left(\frac{\pi k - \mathrm{i}w}{\pi k + \mathrm{i}w}\right)\right] = -\mathrm{i}\frac{\pi k\,\mathrm{e}^{\pi \mathrm{i}k}}{w - \mathrm{i}\pi k}\left[\cos(\pi k) + w\frac{\sin\pi k}{\pi k}\right] \\ &= -\mathrm{i}\frac{\pi k(1 + w)\,\mathrm{e}^{\pi \mathrm{i}k}}{(w - \mathrm{i}\pi k)}\prod_{j=2}^{\infty}\left[1 - \frac{k^2}{\lambda_j^2}\right]. \end{aligned} \tag{3.23}$$

Comparing (3.13) with (3.23), we see that for small w or w', λ_1 approaches zero as

$$\pi\lambda_1 \sim \left(\frac{ww'}{(1+w)(1+w')}\right)^{1/2} \tag{3.24}$$

and one of the zero points of $F_{BB'}(k)$ is cancelled with a pole at $k = -\mathrm{i}w/\pi$ or at $k = -\mathrm{i}w'/\pi$.

3.2. Zeta function regularization

Let us regularize the following divergent products by zeta function regularization:

$$W_{BB'}^{(P)} := \prod_{m\in\mathbb{Z}} \prod_{j=1}^{\infty} \left[\left(\frac{m}{r}\right)^2 + \left(\frac{\pi\lambda_j}{l}\right)^2\right]^{1/2} \tag{3.25}$$

$$W_{BB'}^{(A)} := \prod_{s\in\mathbb{Z}+1/2} \prod_{j=1}^{\infty} \left[\left(\frac{s}{r}\right)^2 + \left(\frac{\pi\lambda_j}{l}\right)^2\right]^{1/2}. \tag{3.26}$$

Similarly, by using the property of $F_{BB'}^{(+)}(k)$, we can obtain the following regularized expressions:

$$\begin{aligned} W_{BB'}^{(P)} &= q^{-(1/2)c_{BB'}^{(+)}} \prod_{j=1}^{\infty}(1-q^{\lambda_j}) \\ &= \frac{\sqrt{8\pi^2 uu'}}{\Gamma(u+1)\Gamma(u'+1)} \left(\frac{u}{e}\right)^u \left(\frac{u'}{e}\right)^{u'} \tilde{q}^{1/24} \prod_{m=1}^{\infty}\left[1+\left(\frac{m-u}{m+u}\right)\left(\frac{m-u'}{m+u'}\right)\tilde{q}^m\right] \end{aligned} \tag{3.27}$$

$$\begin{aligned} W_{BB'}^{(A)} &= q^{-(1/2)c_{BB'}^{(+)}} \prod_{j=1}^{\infty}(1+q^{\lambda_j}) \\ &= \frac{2\pi}{\Gamma(u+1/2)\Gamma(u'+1/2)} \left(\frac{u}{e}\right)^u \left(\frac{u'}{e}\right)^{u'} \\ &\quad \times \tilde{q}^{-1/48} \prod_{m=1}^{\infty}\left[1+\left(\frac{m-\frac{1}{2}-u}{m-\frac{1}{2}+u}\right)\left(\frac{m-\frac{1}{2}-u'}{m-\frac{1}{2}+u'}\right)\tilde{q}^{m-1/2}\right] \end{aligned} \tag{3.28}$$

where

$$c_{BB'}^{(+)} := \int_0^{\infty} \mathrm{d}t\, 2t\, \rho(\mathrm{i}t) \left[1+\mathrm{e}^{2\pi t}\left(\frac{\pi t+w}{\pi t-w}\right)\left(\frac{\pi t+w'}{\pi t-w'}\right)\right]^{-1}. \tag{3.29}$$

Here we have regarded the infinite product

$$\prod_{m=1}^{\infty}\left(1+\frac{u}{m-\frac{1}{2}}\right) = \frac{\Gamma(u+1)\,\mathrm{e}^{-\gamma u}}{\Gamma(2u+1)} \prod_{m=1}^{\infty} \mathrm{e}^{u/(m-(1/2))} \tag{3.30}$$

as

$$\prod_{m=1}^{\infty}\left(1+\frac{u}{m-\frac{1}{2}}\right) = \frac{\sqrt{\pi}}{\Gamma\left(u+\frac{1}{2}\right)}\left(\frac{u}{e}\right)^u \tag{3.31}$$

by replacing the infinite sum by

$$\sum_{m=1}^{\infty}\frac{u}{m-\frac{1}{2}} \to \lim_{s\to 1}\left(u^s\zeta(s,1/2) - \frac{su}{s-1}\right) = \gamma u + u\log u - u + 2u\log 2. \tag{3.32}$$

3.3. Fermionic boundary state

The boundary state for the fermionic sector is defined by relations

$$\begin{aligned}(\partial_2 - \mathrm{i}v)\psi|_{\sigma^1=0}|B(u)\rangle &= \mathrm{i}(\partial_2 + \mathrm{i}v)\tilde{\psi}|_{\sigma^1=0}|B(u)\rangle \\ \langle B(u)|(\partial_2 + \mathrm{i}v)\psi|_{\sigma^1=0} &= -\mathrm{i}\langle B(u)|(\partial_2 - \mathrm{i}v)\tilde{\psi}|_{\sigma^1=0}.\end{aligned} \quad (3.33)$$

The mode expansion is given by

$$\psi(z) = \sum_s \psi_s \frac{1}{\sqrt{r}} \mathrm{e}^{-sz/r} \qquad \tilde{\psi}(\bar{z}) = \sum_s \tilde{\psi}_s \frac{1}{\sqrt{r}} \mathrm{e}^{-s\bar{z}/r}. \quad (3.34)$$

$s \in \mathbb{Z} + \frac{1}{2}$ for *NS* and $s \in \mathbb{Z}$ for *R*, and

$$\{\psi_s, \psi_{s'}\} = \delta_{s+s',0} \qquad \{\tilde{\psi}_s, \tilde{\psi}_{s'}\} = \delta_{s+s',0}. \quad (3.35)$$

The Hamiltonian of this system can be expressed by using the zero-mode generators of the Virasoro algebras:

$$H = \frac{1}{r}\left(L_0 + \tilde{L}_0 - \frac{1}{24}\right). \quad (3.36)$$

For the *NS* sector,

$$L_0 = \sum_{s>0} s\psi_{-s}\psi_s \qquad \tilde{L}_0 = \sum_{s>0} s\tilde{\psi}_{-s}\tilde{\psi}_s \qquad s \in \mathbb{Z} + \tfrac{1}{2} \quad (3.37)$$

and the unique *NS* ground state is normalized as $\langle 0|0\rangle = 1$.

The boundary state for the *NS* sector is given by

$$|B(u)\rangle^{NS} = g_+(u) \exp\left[\sum_{s>0} \mathrm{i}\left(\frac{s-u}{s+u}\right)\psi_{-s}\tilde{\psi}_{-s}\right]|0\rangle \quad (3.38)$$

$${}^{NS}\langle B(u)| = g_+(u)\langle 0| \exp\left[\sum_{s>0} \mathrm{i}\left(\frac{s-u}{s+u}\right)\psi_s\tilde{\psi}_s\right]. \quad (3.39)$$

If we require

$$W^{(A)}_{BB'} = {}^{NS}\langle B(u')|\,\mathrm{e}^{-lH}|B(u)\rangle^{NS} \quad (3.40)$$

the normalization is fixed and we find

$$g_+(u) = \frac{\sqrt{2\pi}}{\Gamma\left(u + \frac{1}{2}\right)}\left(\frac{u}{e}\right)^u. \quad (3.41)$$

For the *R* sector,

$$L_0 = \sum_{m=1}^{\infty} m\psi_{-m}\psi_m + \frac{1}{16} \qquad \tilde{L}_0 = \sum_{m=1}^{\infty} m\tilde{\psi}_{-m}\tilde{\psi}_m + \frac{1}{16} \quad (3.42)$$

and the ground states form the two-dimensional representation of Clifford algebra.

Our convention is

$$\psi_0|\sigma\rangle = \frac{1}{\sqrt{2}}|\mu\rangle \qquad \psi_0|\mu\rangle = \frac{1}{\sqrt{2}}|\sigma\rangle \quad (3.43)$$

$$\tilde{\psi}_0|\sigma\rangle = \frac{\mathrm{i}}{\sqrt{2}}|\mu\rangle \qquad \tilde{\psi}_0|\mu\rangle = -\frac{\mathrm{i}}{\sqrt{2}}|\sigma\rangle \quad (3.44)$$

$$\langle\sigma|\sigma\rangle = \langle\mu|\mu\rangle = 1 \qquad \langle\sigma|\mu\rangle = 0. \quad (3.45)$$

The states $|\sigma\rangle$ and $|\mu\rangle$ correspond to the order field and the disorder field respectively.

Then, it holds that $\psi_0|\sigma\rangle = -\mathrm{i}\tilde{\psi}_0|\sigma\rangle$, $\langle\sigma|\psi_0 = \mathrm{i}\langle\sigma|\tilde{\psi}_0$. Therefore, $|\sigma\rangle$ is chosen to satisfy the condition (3.33).

The boundary state for the R sector is given by

$$|B(u)\rangle^R = g_-(u)\exp\left[\sum_{m=1}^{\infty}\mathrm{i}\left(\frac{m-u}{m+u}\right)\psi_{-m}\tilde{\psi}_{-m}\right]|\sigma\rangle \tag{3.46}$$

$$^R\langle B(u)| = g_-(u)\langle\sigma|\exp\left[\sum_{m=1}^{\infty}\mathrm{i}\left(\frac{m-u}{m+u}\right)\psi_m\tilde{\psi}_m\right]. \tag{3.47}$$

By requiring

$$W_{BB'}^{(P)} = {}^R\langle B(u')|\,\mathrm{e}^{-lH}|B(u)\rangle^R \tag{3.48}$$

we have

$$g_-(u) = \frac{2^{1/4}\sqrt{2\pi u}}{\Gamma(u+1)}. \tag{3.49}$$

Thus, we obtain the same expressions of boundary states and $g_\pm(u)$ as in [16, 17].

Here we summarize the relations as follows:

$$W_{BB'}^{(A)} = \mathrm{Tr}(\mathrm{e}^{-2\pi r H_{BB'}}) = {}^{NS}\langle B(u')|\,\mathrm{e}^{-lH}|B(u)\rangle^{NS} \tag{3.50}$$

$$W_{BB'}^{(P)} = \mathrm{Tr}((-1)^F\,\mathrm{e}^{-2\pi r H_{BB'}}) = {}^R\langle B(u')|\,\mathrm{e}^{-lH}|B(u)\rangle^R. \tag{3.51}$$

In [11], a part of the perturbation terms is identified with the boundary spin operators

$$\sigma_B^{(0)}(\sigma^2) = \omega_0^{-1}a_1(\psi + \mathrm{i}\tilde{\psi})|_{\sigma^1=0} \qquad \sigma_B^{(l)}(\sigma^2) = \omega_0^{-1}a_2(\psi - \mathrm{i}\tilde{\psi})|_{\sigma^1=l}. \tag{3.52}$$

They are nonlocal with respect to the fermionic fields $\psi, \tilde{\psi}$ and yield square root branch points. Thus, the insertion of these operators at the boundaries forms a complete boundary state which is a superposition of the NS and R boundary states:

$$|B;h\rangle := \frac{1}{\sqrt{2}}(|B(u)\rangle^{NS} + \mathrm{sign}(h)|B(u)\rangle^R). \tag{3.53}$$

The fermionic part of the partition function is given by

$$\begin{aligned} Z^{(\psi)}(h,h') &= \tfrac{1}{2}[\mathrm{Tr}(\mathrm{e}^{-2\pi r H_{BB'}}) + \mathrm{sign}(hh')\,\mathrm{Tr}((-1)^F\,\mathrm{e}^{-2\pi r H_{BB'}})] \\ &= \langle B;h'|\,\mathrm{e}^{-lH}|B;h\rangle. \end{aligned} \tag{3.54}$$

Here $B = B(u)$, $B' = B(u')$ and $u = rv = \frac{1}{2}rh^2$, $u' = rv' = \frac{1}{2}r(h')^2$. It is easy to see that even number insertions of σ_B give a nonzero contribution to the partition function.

It is known that the Ising model has three conformally invariant boundary conditions: free (f), fixed up (+) and fixed down (−) [20].

Indeed, the corresponding boundary states are

$$|f\rangle = |B;h=0\rangle \qquad |\pm\rangle = |B;h=\pm\infty\rangle. \tag{3.55}$$

Thus the boundary state (3.53) is intermediate between the free boundary condition and the fixed boundary conditions.

If the boundary magnetic fields h, h' increase from 0 to $\pm\infty$, the open channel partition function (3.54) flows from

$$Z^{(\psi)}(0,0) = Z_{ff} = q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2}) \tag{3.56}$$

to

$$Z^{(\psi)}(\pm\infty, \pm\infty) = Z_{\pm,\pm} = \frac{1}{2}\left[q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2}) + q^{-1/48}\prod_{n=0}^{\infty}(1-q^{n+1/2})\right] \qquad (3.57)$$

$$Z^{(\psi)}(\pm\infty, \mp\infty) = Z_{\pm,\mp} = \frac{1}{2}\left[q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2}) - q^{-1/48}\prod_{n=0}^{\infty}(1-q^{n+1/2})\right]. \qquad (3.58)$$

4. Discussion

In this paper, we have obtained the cylinder partition functions of a few two-dimensional field theory models using the technique of zeta function regularization. (For computation on geometries other than disc and cylinder, see [21].) A subtraction procedure (renormalization) is introduced in order to reproduce the correct expression at the conformal points. From the expression of the partition functions and with the help of the boundary states, the corresponding g-functions are determined. These are main results of this paper.

These results for the partition functions should be proved by using the spectral zeta function for the Laplacian $-\partial^2$. Some works on the zeta function regularization related to the mixed (Robin) boundary condition are found in [22, 23].

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Appendix A. Casimir energies

In this appendix, we rewrite the spectral zeta functions

$$\zeta_{BB'}^{(+)}(s) = \sum_{j=1}^{\infty}\frac{1}{\lambda_j^s} \qquad \zeta_{BB'}^{(-)}(s) = \sum_{j=1}^{\infty}\frac{1}{\nu_j^s} \qquad \mathrm{Re}\, s > 1 \qquad (A.1)$$

in an integral of Hermite type in order to examine their values at $s = -1$.

Naively, Casimir energies $c_{BB'}^{(\pm)}$ were expected to be $\zeta_{BB'}^{(\pm)}(-1)$.

Recall that

$$F_{BB'}^{(\pm)}(k) = \frac{1}{2}\left[1 \pm \mathrm{e}^{2\pi \mathrm{i}k}\left(\frac{\pi k - \mathrm{i}w}{\pi k + \mathrm{i}w}\right)\left(\frac{\pi k - \mathrm{i}w'}{\pi k + \mathrm{i}w'}\right)\right]. \qquad (A.2)$$

Note that

$$\frac{\partial}{\partial k}F_{BB'}^{(\pm)}(k) = 2\pi \mathrm{i}\rho(k)\left[F_{BB'}^{(\pm)}(k) - \frac{1}{2}\right] \qquad (A.3)$$

where

$$\rho(k) = 1 + \frac{w}{\pi^2k^2 + w^2} + \frac{w'}{\pi^2k^2 + (w')^2} \qquad (A.4)$$

$$\frac{1}{2F_{BB'}^{(\pm)}(k)} + \frac{1}{2F_{BB'}^{(\pm)}(-k)} = 1. \qquad (A.5)$$

Collecting these relations, we have

$$U(k) := \frac{\partial}{\partial k}\log F^{(\pm)}_{BB'}(k) = 2\pi\mathrm{i}\rho(k)\left[1 - \frac{1}{2F^{(\pm)}_{BB'}(k)}\right] = 2\pi\mathrm{i}\rho(k)\frac{1}{2F^{(\pm)}_{BB'}(-k)}. \tag{A.6}$$

Explicitly, it can be written as

$$\frac{\partial}{\partial k}\log F^{(+)}_{BB'}(k) = \mathrm{i}\pi - \frac{\pi}{\pi k + \mathrm{i}w} - \frac{\pi}{\pi k + \mathrm{i}w'} + \sum_{j=1}^{\infty}\left(\frac{1}{k-\lambda_j} + \frac{1}{k+\lambda_j}\right) \tag{A.7}$$

$$\frac{\partial}{\partial k}\log F^{(-)}_{BB'}(k) = \mathrm{i}\pi + \frac{1}{k} - \frac{\pi}{\pi k + \mathrm{i}w} - \frac{\pi}{\pi k + \mathrm{i}w'} + \sum_{j=1}^{\infty}\left(\frac{1}{k-\nu_j} + \frac{1}{k+\nu_j}\right). \tag{A.8}$$

Let $\mu_j = \lambda_j$ (ν_j) for $+(-)$. For a natural number M, let us choose a number N such that $\mu_M < N < \mu_{M+1}$.

For simplicity, we assume $w, w' > 0$. Then $\mu_1 > 0$ and we can choose a real number δ such that $0 < \delta < \mu_1$.

Let a union of segments of the real axis I be

$$I = I_0 \cup I_1 \cup \cdots \cup I_M \tag{A.9}$$

where

$$I_0 = [\delta, \mu_1 - \epsilon] \qquad I_j = [\mu_j + \epsilon, \mu_{j+1} - \epsilon] \qquad (j = 1, \ldots, M-1)$$
$$I_M = [\mu_M + \epsilon, N]. \tag{A.10}$$

Here ϵ is an infinitesimally small positive number.

For an analytic function $W(k)$ bounded on the strip $0 \leqslant \mathrm{Re}\, k \leqslant N$, we have

$$0 = \int_{C_1} \mathrm{d}k\, U(k)W(k). \tag{A.11}$$

The integration contour C_1 is shown in figure 1. We get

$$\begin{aligned}0 = \int_I \mathrm{d}k\, U(k)W(k) + \mathrm{i}\int_0^R \mathrm{d}t\, U(N+\mathrm{i}t)W(N+\mathrm{i}t) - \mathrm{i}\int_\delta^R \mathrm{d}t\, U(\mathrm{i}t)W(\mathrm{i}t)\\ - \mathrm{i}\int_0^{\pi/2} \mathrm{d}\theta\, \delta\, \mathrm{e}^{\mathrm{i}\theta} U(\delta\,\mathrm{e}^{\mathrm{i}\theta})W(\delta\,\mathrm{e}^{\mathrm{i}\theta}) - \int_0^N \mathrm{d}t\, U(\mathrm{i}R + t)W(\mathrm{i}R + t)\\ - \sum_{j=1}^{M} \mathrm{i}\int_0^{\pi} \mathrm{d}\theta\, \epsilon\, \mathrm{e}^{\mathrm{i}\theta} U(\mu_j + \epsilon\,\mathrm{e}^{\mathrm{i}\theta})W(\mu_j + \epsilon\,\mathrm{e}^{\mathrm{i}\theta}).\end{aligned} \tag{A.12}$$

If we take the $R \to \infty$ limit, we have

$$\begin{aligned}\mathrm{i}\pi \sum_{j=1}^{M} W(\mu_j) + \mathcal{O}(\epsilon) = \int_I \mathrm{d}k\, U(k)W(k) - \mathrm{i}\int_\delta^\infty \mathrm{d}t\, U(\mathrm{i}t)W(\mathrm{i}t)\\ + \mathrm{i}\int_0^\infty \mathrm{d}t\, U(N+\mathrm{i}t)W(N+\mathrm{i}t) - \mathrm{i}\int_0^{\pi/2} \mathrm{d}\theta\, \delta\, \mathrm{e}^{\mathrm{i}\theta} U(\delta\,\mathrm{e}^{\mathrm{i}\theta})W(\delta\,\mathrm{e}^{\mathrm{i}\theta}).\end{aligned} \tag{A.13}$$

Similarly, from

$$0 = \int_{C_2} \mathrm{d}k\, U(-k)W(k) \tag{A.14}$$

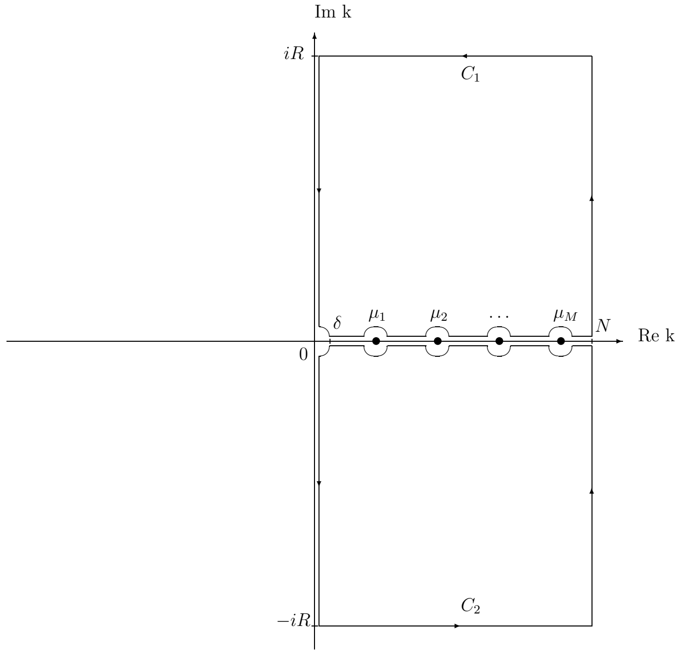

Figure 1. Contours of integration.

(see figure 1 for the integration contour C_2), we have

$$\mathrm{i}\pi\sum_{j=1}^{M}W(\mu_j)+\mathcal{O}(\epsilon)=\int_I \mathrm{d}k\,U(-k)W(k)+\mathrm{i}\int_\delta^\infty \mathrm{d}t\,U(\mathrm{i}t)W(-\mathrm{i}t)$$
$$-\,\mathrm{i}\int_0^\infty \mathrm{d}t\,U(-N+\mathrm{i}t)W(N-\mathrm{i}t)-\mathrm{i}\int_0^{\pi/2}\mathrm{d}\theta\,\delta\,\mathrm{e}^{-\mathrm{i}\theta}U(-\delta\,\mathrm{e}^{-\mathrm{i}\theta})W(\delta\,\mathrm{e}^{-\mathrm{i}\theta}). \tag{A.15}$$

Adding (A.13) and (A.15) gives

$$\sum_{j=1}^{M}W(\mu_j)=\int_\delta^N \mathrm{d}k\,\rho(k)W(k)-\frac{1}{2\pi}\int_\delta^\infty \mathrm{d}t\,U(\mathrm{i}t)\,[W(\mathrm{i}t)-W(-\mathrm{i}t)]$$
$$+\frac{1}{2\pi}\int_0^\infty \mathrm{d}t[U(N+\mathrm{i}t)W(N+\mathrm{i}t)-U(-N+\mathrm{i}t)W(N-\mathrm{i}t)]$$
$$-\frac{1}{2\pi}\delta\int_0^{\pi/2}\mathrm{d}\theta[\mathrm{e}^{\mathrm{i}\theta}U(\delta\,\mathrm{e}^{\mathrm{i}\theta})W(\delta\,\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{-\mathrm{i}\theta}U(-\delta\,\mathrm{e}^{-\mathrm{i}\theta})W(\delta\,\mathrm{e}^{-\mathrm{i}\theta})]. \tag{A.16}$$

Here we use a relation

$$U(k)+U(-k)=2\pi\mathrm{i}\rho(k) \tag{A.17}$$

and take the $\epsilon\to 0$ limit.

Let us set

$$W(k)=k^{-s}=|k|^{-s}\,\mathrm{e}^{-s\arg(k)}\qquad -\pi<\arg(k)<\pi. \tag{A.18}$$

Then for $\mathrm{Re}\, s > 1$, we have (as $M \to \infty$)

$$\zeta_{BB'}^{(\pm)}(s) = \int_\delta^\infty \mathrm{d}k\, \rho(k) \frac{1}{k^s} - 2\sin\Big(\frac{\pi}{2}s\Big) \int_\delta^\infty \mathrm{d}t \frac{t^{-s}}{1 \pm \mathrm{e}^{2\pi t} \left(\frac{\pi t + w}{\pi t - w}\right)\left(\frac{\pi t + w'}{\pi t - w'}\right)} \rho(\mathrm{i}t)$$
$$- \frac{1}{2\pi} \delta^{1-s} \int_0^{\pi/2} \mathrm{d}\theta [\mathrm{e}^{-\mathrm{i}(s-1)\theta} U(\delta\, \mathrm{e}^{\mathrm{i}\theta}) + \mathrm{e}^{\mathrm{i}(s-1)\theta} U(-\delta\, \mathrm{e}^{-\mathrm{i}\theta})]. \qquad (A.19)$$

The second and the third terms on the right-hand side of equation (A.19) make sense in the whole complex s-plane. When $\delta \to 0$, the third term vanishes for $\mathrm{Re}\, s < 0$.

Let us consider the first term,

$$\int_\delta^\infty \mathrm{d}k\, \rho(k) k^{-s} = \frac{1}{s-1} \delta^{-s+1} + V(\delta, w, s) + V(\delta, w', s). \qquad (A.20)$$

Here

$$V(\delta, w, s) = \int_\delta^\infty \mathrm{d}k \frac{1}{k^s} \frac{w}{\pi^2 k^2 + w^2}. \qquad (A.21)$$

This integral makes sense for $\mathrm{Re}\, s > -1$.

Note that for $\mathrm{Re}\, s > -1$, $V(\delta, 0, s) = 0$, $V(\delta, \infty, s) = 0$.

For $0 < w < \infty$, it is possible to change the integration variable from k to $t = w^2/(\pi^2 k^2 + w^2)$, and then $V(\delta, w, s)$ becomes

$$V(\delta, w, s) = \frac{1}{2\pi} \left(\frac{\pi}{w}\right)^s \int_0^\eta \mathrm{d}t\, t^{(s-1)/2} (1-t)^{-(s+1)/2}$$
$$= \frac{1}{\pi(s+1)} \left(\frac{\pi}{w}\right)^s \eta^{(s+1)/2} F\left(\frac{s+1}{2}, \frac{s+1}{2}; \frac{s+3}{2}; \eta\right) \qquad (A.22)$$

where $\eta = w^2/(\pi^2 \delta^2 + w^2)$ and $F(\alpha, \beta; \gamma; \eta)$ is the hypergeometric function. With the help of properties of the hypergeometric function, we can see that $V(\delta, w, s)$ has simple poles at

$$s = 1 - 2m \qquad m = 1, 2, 3, \ldots. \qquad (A.23)$$

For $-3 < \mathrm{Re}\, s < 1$, $\delta \to 0$ limit gives

$$V(0, w, s) = \frac{1}{2\sin((s+1)\pi/2)} \left(\frac{\pi}{w}\right)^s. \qquad (A.24)$$

Thus we conclude that $\zeta_{BB'}^{(\pm)}(-1)$ diverges for $0 < w, w' < \infty$.

We drop these divergent terms by hand and assume that the Casimir energies are given by

$$c_{BB'}^{(\pm)} := \int_0^\infty \mathrm{d}t\, 2t\, \rho(\mathrm{i}t) \left[1 \pm \mathrm{e}^{2\pi t} \left(\frac{\pi t + w}{\pi t - w}\right)\left(\frac{\pi t + w'}{\pi t - w'}\right)\right]^{-1}. \qquad (A.25)$$

These are finite and give correct values at conformal points. Indeed,

$$\int_0^\infty \mathrm{d}t \frac{2t}{1 + \mathrm{e}^{2\pi t}} = \frac{1}{24} \qquad \int_0^\infty \mathrm{d}t \frac{2t}{1 - \mathrm{e}^{2\pi t}} = -\frac{1}{12}. \qquad (A.26)$$

Integrating by parts, we can see that the above expressions of the Casimir energies are consistent with other expressions [9, 16].

For SBQD model, $\zeta_{BB'}^{(\pm)}(-1)$ appears only in a factor $q^{(\zeta_{BB'}^{(-)}(-1) - \zeta_{BB'}^{(+)}(-1))/2}$. In this case, the divergent terms cancel each other:

$$\lim_{s \to -1} \left(\zeta_{BB'}^{(-)}(s) - \zeta_{BB'}^{(+)}(s)\right) = c_{BB'}^{(-)} - c_{BB'}^{(+)}. \qquad (A.27)$$

So we do not need to discard the divergent terms.

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